Numerical Methods ITGS219

Lecture 4 - Taylor Series

By: Zahra Elashaal

What is a Taylor series?

The Taylor series of a function is an infinite sum of terms that are expressed in terms of the function's derivatives at a single point.

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots,$$

this can be written as: $$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

Some examples of Taylor series which you must have seen

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

The Maclaurin series for the exponential function e^x is:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$
$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots.$$

The Maclaurin series for $1/(1 - x)$ is the geometric series

$$1 + x + x^2 + x^3 + \cdots,$$

The corresponding Taylor series for $\ln x$ at a = 1 is

$$(x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \tfrac{1}{4}(x-1)^4 + \cdots,$$

All the above expressions are actually a special case of Taylor series called the Maclaurin series.

General Taylor Series

The general form of the Taylor series is given by

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3 + \cdots$$

provided that all derivatives of *f(x)* are continuous and exist in the interval [*x*, *x*+*h*]

What does this mean in plain English?

As **Archimedes** would have said, "Give me the value of the function at a single point, and the value of all (first, second, and so on) its derivatives at that single point, and I can give you the value of the function at any other point"

Example 1—Taylor Series

Find the value of $f(6)$ given that $f(4)=125$, $f'(4)=74$, $f''(4)=30$, $f'''(4)=6$ and all other higher order derivatives of $f(x)$ at $x=4$ are zero.

Solution:

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2!} + f'''(x)\frac{h^3}{3!} + \cdots$$

$x = 4 \qquad h = 6 - 4 = 2$

Since the higher order derivatives are zero,

$$f(4+2) = f(4) + f'(4)2 + f''(4)\frac{2^2}{2!} + f'''(4)\frac{2^3}{3!}$$

$$f(6) = 125 + 74(2) + 30\left(\frac{2^2}{2!}\right) + 6\left(\frac{2^3}{3!}\right)$$

$$= 125 + 148 + 60 + 8 = 341$$

Note that to find $f(6)$ exactly, we only need the value of the function and all its derivatives at some other point, in this case $x = 4$

Example 2

Find the value of e^x using the first five terms of the Maclaurin series at $x=0.25$.

Solution

The first five terms of the Maclaurin series for is

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$e^{0.25} \approx 1 + 0.25 + \frac{0.25^2}{2!} + \frac{0.25^3}{3!} + \frac{0.25^4}{4!}$$

$$= 1.2840$$

The exact value of $e^{0.25}$ up to 5 significant digits is also 1.2840.

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Derivation for Maclaurin Series for e^x

Derive the Maclaurin series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

The Maclaurin series is simply the Taylor series about the point x=0

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2!} + f'''(x)\frac{h^3}{3!} + f''''(x)\frac{h^4}{4} + f'''''(x)\frac{h^5}{5} + \cdots$$

$$f(0+h) = f(0) + f'(0)h + f''(0)\frac{h^2}{2!} + f'''(0)\frac{h^3}{3!} + f''''(0)\frac{h^4}{4} + f'''''(0)\frac{h^5}{5} + \cdots$$

Since $f(x) = e^x$, $f'(x) = e^x$, $f''(x) = e^x$, ..., $f^n(x) = e^x$

and $f^n(0) = e^0 = 1$

the Maclaurin series is then

$$f(h) = (e^0) + (e^0)h + \frac{(e^0)}{2!}h^2 + \frac{(e^0)}{3!}h^3 \ldots$$

$$= 1 + h + \frac{1}{2!}h^2 + \frac{1}{3!}h^3 \ldots$$

So,

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

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The Error in Taylor Series

The Taylor polynomial of order n of a function $f(x)$ with $(n+1)$ continuous derivatives in the domain $[x, x+h]$ is given by

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2!} + \cdots + f^{(n)}(x)\frac{h^n}{n!} + R_n(x)$$

where the remainder is given by

$$R_n(x+h) = \frac{(h)^{n+1}}{(n+1)!} f^{(n+1)}(c) \quad ; \quad x < c < x+h$$

that is, c is some point in the domain [x, x+h]

Example—error in Taylor series

The Taylor series for e^x at point $x=0$ is given by:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

It can be seen that as the number of terms used increases, the error bound decreases and hence a better estimate of the function can be found.

Example ... cont.

How many terms would it require to get an approximation of e^1 within a magnitude of true error of less than 10^{-6}.

Solution:

Using (n+1) terms of Taylor series gives error bound of

$$R_n(x+h) = \frac{(h)^{n+1}}{(n+1)!} f^{(n+1)}(c) \; ; \; x = 0, h = 1, f(x) = e^x$$

$$R_n(0+1) = \frac{(1)^{n+1}}{(n+1)!} f^{(n+1)}(c) = \frac{(1)^{n+1}}{(n+1)!} e^c$$

Since

$$x < c < x+h$$
$$0 < c < 0+1$$
$$0 < c < 1$$

$$\frac{1}{(n+1)!} < |R_n(0)| < \frac{e}{(n+1)!}$$

So if we want to find out how many terms it would require to get an approximation of e^1 within a magnitude of true error of less than 10^{-6}

$$\frac{e}{(n+1)!} < 10^{-6}$$
$$(n+1)! > 10^6 e$$
$$(n+1)! > 10^6 \times 3$$
$$n \geq 9$$

So 9 terms or more are needed to get a true error less than 10^{-6}


Any Question?