Numerical Methods ITGS219

Lecture: Interpolation and Extrapolation

Splines

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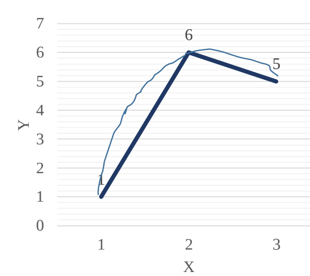
5.6 Splines



Another form of interpolation: this fits a curve in an interval where the data at the end of the interval **coincides** with the data points and also **matches** the available derivatives with the next interval.

If we consider the points (x_i, f_i) and (x_{i+1}, f_{i+1}) and fit a straight line this will match the function values with the intervals on each side (but not necessarily the derivative).

We need to use a **higher-order curve**. In order to satisfy the requirements we need at least a **cubic**, and this is often the preferred option.

If we consider the **points** to be **(x_i, f_i)** for ***$i = 1$ to N***, we have *$N-1$ intervals*. Let us consider the cubic valid over the interval [x_i, x_{i+1}] to be:

$$y_i(x) = a_i + (x - x_i)b_i + (x - x_i)^2 c_i + (x - x_i)^3 d_i.$$

We note that we have ***$N-1$ cubic equations*** each with four unknowns; that is ***unknowns = $4N-4$*** in total.

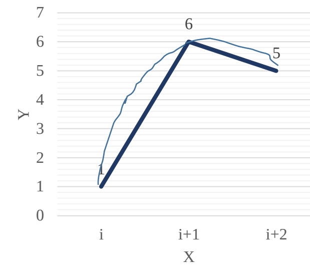
5.6 Splines … cont.

The details of **the conditions** used to specify the values of the coefficients. Or to get the equations for ***unknowns = 4N−4*** equations

1) we require that the cubic matches the data values at the ends of the interval:

$$y_i(x_i) = f_i \text{ and } y_i(x_{i+1}) = f_{i+1} \text{ for } i = 1, \cdots, N-1;$$

this in effect fixes the a_i and yields *2(N−1)* equations.



2) we require the gradient and the curvature to match at the interior points:

$$y_i'(x_{i+1}) = y_{i+1}'(x_{i+1}) \text{ and } y_i''(x_{i+1}) = y_{i+1}''(x_{i+1}) \text{ for } i = 1, \cdots, N-2.$$

This yields a further *2(N−2)* equations. Unfortunately we only have *4N−6* equations and as such our system is not totally specified.

3) We elect to consider **natural splines**, for which the curvature is taken to be **zero** at the ends of the domain, that is

$$y_1''(x_1) = 0 \text{ and } y_{N-1}''(x_N) = 0.$$

So that we get the *4N−4* equations

5.6 Cubic Splines Example

Example Construct a natural cubic spline that passes through the points

	i	*i+1*	*i+2*
x	1	2	3
y	2	3	5

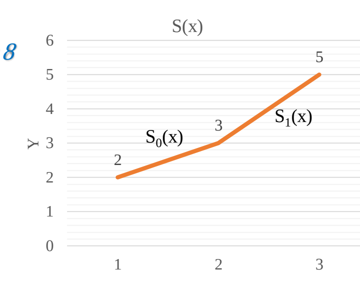
$$y_i(x) = a_i + (x - x_i)b_i + (x - x_i)^2 c_i + (x - x_i)^3 d_i.$$

cubic equations = N−1 = 3-1 = 2

[1,2] ; $S_0(x) = a_0 + b_0 (x-1) + c_0 (x-1)^2 + d_0 (x-1)^3$

[2,3] ; $S_1(x) = a_1 + b_1 (x-2) + c_1 (x-2)^2 + d_1 (x-2)^3$

8 unknowns *4(N−1) =4(3-1)= 8*



1) cubic matches the data values at the ends

$y_i(x_i)=f_i$ and $y_i(x_{i+1})=f_{i+1}$ for $i=1,\dots,N-1$

gives 2(N−1) =2(3-1)= 4 equations

$S_0(1) = 2$ => $a_0 = 2$ 7 unknowns to go

$S_0(2) = 3$ => $a_0 + b_0 + c_0 + d_0 = 3$ => $b_0 + c_0 + d_0 = 1$ ----(1)

$S_1(2) = 3$ => $a_1 = 3$ 6 unknowns to go

$S_1(3) = 5$ => $3 + b_1 + c_1 + d_1 = 5$ => $b_1 + c_1 + d_1 = 2$ ----(2)

5.6 Cubic Splines Example

	i	$i+1$	$i+2$
x	1	2	3
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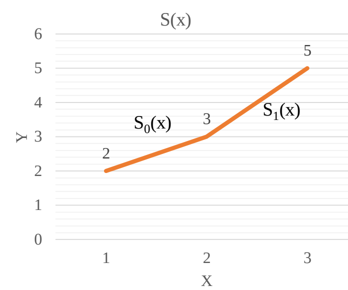
Example Construct a natural cubic spline that passes through the points

$$y_i(x) = a_i + (x - x_i)b_i + (x - x_i)^2 c_i + (x - x_i)^3 d_i.$$

[1,2] ; $S_0(x) = a_0 + b_0 (x\text{-}1) + c_0 (x\text{-}1)^2 + d_0 (x\text{-}1)^3$

[2,3] ; $S_1(x) = a_1 + b_1 (x\text{-}2) + c_1 (x\text{-}2)^2 + d_1 (x\text{-}2)^3$



2) we require the gradient and the curvature to match at the interior points:

$y'_i(x_{i+1}) = y'_{i+1}(x_{i+1})$ and $y''_i(x_{i+1}) = y''_{i+1}(x_{i+1})$ for $i=1,\ldots,N\text{-}2$

gives $2(N-2) = 2(3\text{-}2) = 2$ equations

$S_0'(x) = b_0 + 2c_0 (x\text{-}1) + 3d_0 (x\text{-}1)^2 \quad \Rightarrow \quad S_0''(x) = 2c_0 + 6d_0 (x\text{-}1)$

$S_1'(x) = b_1 + 2c_1 (x\text{-}2) + 3d_1 (x\text{-}2)^2 \quad \Rightarrow \quad S_1''(x) = 2c_1 + 6d_1 (x\text{-}2)$

$S_0'(2) = S_1'(2) \quad \Rightarrow \quad b_0 + 2c_0 + 3d_0 = b_1$ ----(3)

$S_0''(2) = S_1''(2) \quad \Rightarrow \quad 2c_0 + 6d_0 = 2c_1$ ----(4)

5.6 Cubic Splines Example

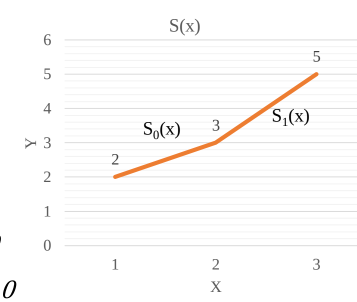
x	1	2	3
y	2	3	5

3) We elect to consider **natural splines**, for which the curvature is taken to be **zero** at the ends of the domain, that is

Boundary Condition $\quad y''_1(x_1) = 0$ and $y''_{N-1}(x_N) = 0.$

$S_0''(1) = 0 \quad \Rightarrow \quad c_0 = 0$ 5 unknowns to go

And $S_1''(3) = 0 \quad \Rightarrow \quad 2c_1 + 6d_1 = 0$ ----(5)



$b_0 + d_0 = 1$; $b_1 + c_1 + d_1 = 2$; $b_0 + 3d_0 = b_1$; $3d_0 = c_1$; $2c_1 + 6d_1 = 0$

$b_0 + d_0 = 1$; $b_1 + c_1 + d_1 = 2$; $b_0 + 3d_0 - b_1 = 0$; $3d_0 - c_1 = 0$; $2c_1 + 6d_1 = 0$

$$\begin{array}{ccccc} b_0 & d_0 & b_1 & c_1 & d_1 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 3 & -1 & 0 & 0 \\ 0 & 3 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$b_0 = 3/4 \quad d_0 = 1/4 \quad b_1 = 3/2 \quad c_1 = 3/4 \quad d_1 = -¼ \quad a_0 = 2 \quad a_1 = 3 \quad c_0 = 0$

$a_0 = 2 \quad b_0 = 3/4 \quad c_0 = 0 \quad d_0 = ¼$

$a_1 = 3 \quad b_1 = 3/2 \quad c_1 = 3/4 \quad d_1 = -¼$

$$S(x) = \begin{cases} 2 + \frac{3}{4}(x-1) + \frac{1}{4}(x-1)^3 & ;\ x \in [1,2] \\ 3 + \frac{3}{2}(x-2) + \frac{3}{4}(x-2)^2 - \frac{1}{4}(x-2)^3 & ;\ x \in [2,3] \end{cases}$$

5.6 Cubic Splines

x	1	2	3
y	2	3	5

Example Same Previous problem but instead of natural cubic spline lets fine the Clamped Cubic Spline if $S'(1) = 2$ and $S'(3) = 1$

$$S_0'(x) = b_0 + 2c_0(x-1) + 3d_0(x-1)^2$$
$$S_1'(x) = b_1 + 2c_1(x-2) + 3d_1(x-2)^2$$

Clamped Conditionals

$S_0'(1) = 2$ => $b_0 = 2$

And $S_1'(3) = 1$ => $b_1 + 2c_1 + 3d_1 = 1$

$$S(x) = \begin{cases} 2 + \frac{3}{4}(x-1) + \frac{1}{4}(x-1)^3 & ; x \in [1,2] \\ 3 + \frac{3}{2}(x-2) + \frac{3}{4}(x-2)^2 - \frac{1}{4}(x-2)^3 & ; x \in [2,3] \end{cases}$$

$a_0 = 2$; $b_0 + c_0 + d_0 = 1$; $a_1 = 3$; $b_1 + c_1 + d_1 = 2$; $b_0 + 2c_0 + 3d_0 = b_1$; $2c_0 + 6d_0 = 2c_1$

$a_0 = 2$; $c_0 + d_0 = -1$; $a_1 = 3$; $b_1 + c_1 + d_1 = 2$; $b_0 + 2c_0 + 3d_0 - b_1 = -2$; $2c_0 + 6d_0 = 2c_1$

$$S(x) = \begin{cases} 2 + 2(x-1) - \frac{5}{2}(x-1)^2 + \frac{3}{2}(x-1)^3 & ; x \in [1,2] \\ 3 + \frac{3}{2}(x-2) + 2(x-2)^2 - \frac{3}{2}(x-2)^3 & ; x \in [2,3] \end{cases}$$

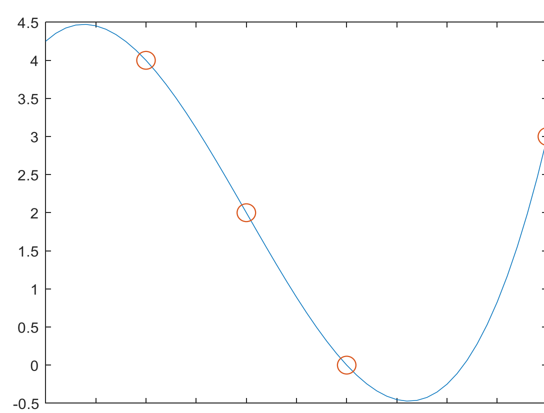
5.6 Splines with MATLAB

We can construct a matrix system in order to determine the coefficients. This can be coded: however the algorithm is quite involved and again Matlab comes to our rescue with the command ***spline***.

Example 5.4 Fit a **cubic spline** to the data x = (1, 2, 3, 5), f = (4, 2, 0, 3) and plot the interpolated function on a grid z = 0, 0.1, 0.2, · · · , 5.

```
x = [1 2 3 5];
f = [4 2 0 3];
z = 0:0.1:5;
y = spline(x,f,z);
plot(z,y,x,f,'o','MarkerSize',12)
```

This yields the next figure



5.6 Splines with MATLAB

By way of illustration we shall now *construct* the *splines* through the *points* $(x_1, y_1), (x_2, y_2), (x_3, y_3)$. These curves are taken to have the equations.

$$y_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3 \quad ; \qquad \text{where j = 1, 2 and 3.}$$

This gives us 12 undetermined coefficients and we shall now describe the equations which they satisfy.

$$y_j(x_j) = f_j \quad , j = 1, 2, 3:$$

This represents the condition that the spline goes through the data point at its left-hand end and these give the equations:

$$a_1 = f_1, \quad (5.3a)$$
$$a_2 = f_2, \quad (5.3b)$$
$$a_3 = f_3. \quad (5.3c)$$

$$y_j(x_{j+1}) = f_{j+1} \quad , j = 1, 2, 3:$$

This gives a similar condition at the right-hand ends:

$$a_1 + h_1 b_1 + h_1^2 c_1 + h_1^3 d_1 = f_2, \quad (5.3d)$$
$$a_2 + h_2 b_2 + h_2^2 c_2 + h_2^3 d_2 = f_3, \quad (5.3e)$$
$$a_3 + h_3 b_3 + h_3^2 c_3 + h_3^3 d_3 = f_4. \quad (5.3f)$$

Here we have introduced $h_j = x_{j+1} - x_j$ for convenience

$$y'_j(x_{j+1}) = y'_{j+1}(x_{j+1}) \quad ; \text{ j = 1, 2:}$$

This is the matching of the first derivative at the internal points,

5.6 Splines with MATLAB

This is the matching of the first derivative at the internal points, which gives

$$b_1 + 2h_1 c_1 + 3h_1^2 d_1 = b_2, \quad (5.3g)$$
$$b_2 + 2h_2 c_2 + 3h_2^2 d_2 = b_3. \quad (5.3h)$$

$y''_j(x_{j+1}) = y''_{j+1}(x_{j+1})$, j = 1, 2: This is a similar condition for the second derivative at the internal points:

$$2c_1 + 6h_1 d_1 = c_2, \quad (5.3i)$$
$$2c_2 + 6h_2 d_2 = c_3. \quad (5.3j)$$

$y''_1(x_1) = 0$ and $y''_3(x_4) = 0$

These are the conditions that the *splines* at the *ends* of the domain are *linear* at the end points, which gives:

$$c_1 = 0, \quad (5.3k)$$
$$2c_3 + 3h_3 d_3 = 0. \quad (5.3l)$$

The solution of the above equations gives the required coefficients. Notice that these coefficients could also have been retrieved using the ***spline*** command and then using the elements of the answer ***coefs***

```
pp = spline(x,f);
pp.coefs
```

Note that the coefficients correspond to a cubic without the $-x_j$ factors. This is an example of an object rather than a variable, that is pp.

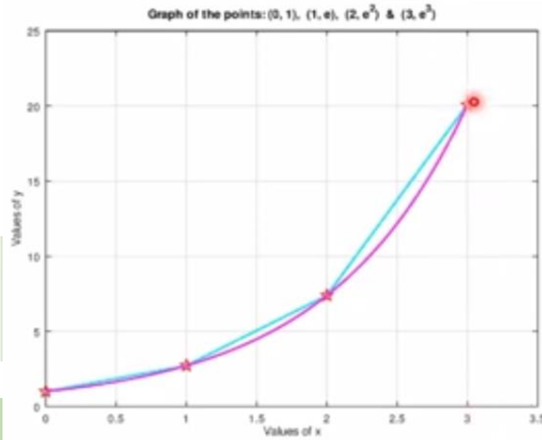
ans =

a_i	b_i	c_i	d_i
0.2917	-0.8750	-1.4167	4.0000
0.2917	0	-2.2917	2.0000
0.2917	0.8750	-1.4167	0

5.6 Splines

Example from the following table approximate *f(1.5)*

x	0	1	2	3
y	1	e	e^2	e^3



The coefficients of the spline on the subintervals are:

a(i)	b(i)	c(i)	d(i)
1.00000000	1.00000000	0.44468250	0.27359933
2.71828183	2.71016299	1.26548049	0.69513079
7.38905610	7.32651634	3.35087286	2.01909162

The clamed spline is:

$$S(x) = \begin{cases} 1 + x + 0.44468\,x^2 + 0.27360\;x^3, & if\ 0 \le x < 1 \\ 2.71828 + 2.71016\,(x-1) + 1.26548(x-1)^2 + 0.69513\,(x-1)^3, & if\ 1 \le x < 2 \\ 7.38906 + 7.32652\,(x-2) + 3.35087(x-2)^2 + 2.01909\,(x-2)^3\ , & if\ 2 \le x \le 3 \end{cases}$$

Now, $S(1.5) = 2.71828 + 2.71016\,(1.5-1) + 1.26548(1.5-1)^2 + 0.69513\,(1.5-1)^3$

$= 4.47662$

Any Question?